7. THE WEDDERBURN STRUCTURE THEOREM

§7.1. The Density Theorem

Theorem 1: Suppose R is a primitive ring with DCC on right ideals.
Then $R \cong M_n(D)$ for some n and some division ring D.
Proof: Let M be a faithful irreducible R-module.
Then $D = End_R(M)$ is a division ring.
Let $n = \dim_D M < \infty$.
Now R is isomorphic to a subring of $End_D(M)$, which is isomorphic to $M_n(D)$.
It remains to show that this map is onto, in other words, that every endomorphism of M is equivalent to right multiplication by some element of R.

Let $f \in End_D(M)$ and let $v_1, \ldots, v_n$ be a basis for M.
For each i let V_i be the subspace spanned by all these basis vectors excluding v_i.

By Theorem 5 it follows that for each i there exists $t_i \in R$ such that $V_i t_i = 0$ and $v_i t_i \neq 0$.

Since M is irreducible, $v_i t_i R = M$. Thus there exists s_i such that $v_i t_i s_i = v_i f$.
Then $t_i s_i \in A(V_i)$.

So for all i, $v_i(t_1s_1 + ... + t_ns_n) = v_if$.
Hence f is the image of $t_1s_1 + ... + t_ns_n$.
Right multiplication by Σt_is_i is thus equivalent to the application of f. ✋☺

§7.2. The Wedderburn Structure Theorem

In the following theorems 'matrix ring' is to be interpreted as the ring of all $n \times n$ matrices over a division ring.

Theorem 2: Every nil-semisimple ring R with DCC on right ideals is isomorphic to a direct sum of matrix rings over division rings.
Proof: By the descending chain condition there exists a minimal right ideal S, and S is a faithful R/A(S) module.

Let T = RS + S. Then T is a 2-sided ideal and A(T) = A(S).

If SR = 0 then $R \leq A(S) = A(T)$ whence TR = 0 and so $T^2 = 0$, a contradiction.

So S is a non-trivial module and hence is a faithful irreducible R/A(S) module.

This means that R/A(T) is a primitive ring with DCC on right ideals.

So $R/A(T) \cong M_n(D)$ for some n and some division ring D, and in particular it is simple. Thus T+A(T) = A(T) or R.

If T + A(T) = A(T) then $T \leq A(T)$ whence $T^2 = 0$ and so T = 0, a contradiction. Hence T + A(T) = R.

Let $M = T \cap A(T)$. Then $M^2 = 0$ and so M = 0.
Thus $R = T \oplus A(T)$ and $T \cong R/A(T) \cong M_n(D)$.
Continuing in this way we obtain the result. ✋☺
Corollary: Nil-semisimple rings with DCC on right ideals have a multiplicative identity.

Theorem 3: A division ring D can't be properly finite-dimensional over an algebraically closed field F.
Proof: Suppose $\dim_F D = n < \infty$ and let $d \in D - F$.
Now $1, d, d^2, \ldots, d^n$ are linearly dependent and so $f(d) = 0$ for some monic polynomial $f(x)$ of degree at most n, with coefficients in F.

Since F is algebraically closed:

$$f(x) = (x - \lambda_1) \ldots (x - \lambda_n) \text{ for some } \lambda_i\text{'s} \in F.$$

Since f(d) = 0, $d \in F$, a contradiction. ✋☺

Theorem 4: If R is a finite-dimensional nil-semisimple algebra over an algebraically closed field F then R is isomorphic to a direct sum of matrix rings over F. That is:

$$R \cong M_{n_1}(F) \oplus M_{n_2}(F) \oplus \ldots \oplus M_{n_k}(F)$$

for some $n_1, n_2, \ldots, n_k$.

Proof: The division rings in the Wedderburn theorem are finite dimensional over F and so are equal to F. ✋☺

Theorem 5: For rings with DCC on right ideals the Jacobson radical coincides with the nil radical.
Proof: If a ring is nil-semisimple and has DCC on right ideals then it is isomorphic to a direct sum of matrix rings over division rings. These are Jacobson semisimple. ✋☺

§7.3. Unitary Modules

Theorem 6: Every submodule N of a unitary module M, over a nil-semisimple ring R with DCC on right ideals, is a direct summand.
Proof: By Wedderburn R is a direct sum of matrix rings over division rings and so

$$R = R_1 \oplus R_2 \oplus \ldots \oplus R_k$$

where each R_i is a minimal right ideal.

Let S be the set of all $X \subseteq M$ such that:

(1) $XR \cap N = 0$;
(2) $XR = \oplus\Sigma x_i R$ for $x_i \in X$;
(3) xR is irreducible for all $x \in X$.

By Zorn's Lemma, S has a maximal element, say X_0.
Then $X_0R \cap N = 0$ and $X_0R = \oplus\Sigma xR$ for $x \in X_0$ and so is a direct sum of irreducible submodules.

Put $K = X_0R$. Suppose $K \oplus N < M$.
Let $m \in M - (K \oplus N)$.

Then $mR = \oplus\Sigma mR_i$.

Since M is unitary $m \in mR$.
Hence for some i, mR_i is not a subset of $K \oplus N$.

By the minimality of R_i, mR_i is irreducible and so

$$mR_i = m'R \text{ for some } m' \in M.$$

Hence $m'R \cap (K \oplus N) = 0$.
Thus $(K \oplus m'R) \cap N = 0$ and so $X_0 \cup \{m'\} \in S$, a contradiction. Thus $M = N \oplus K$.
[Note that K is a direct sum of irreducible submodules.]
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Corollary: Every unitary module over a nil-semisimple ring with DCC on right ideals is a direct sum of irreducible submodules.
Proof: Put $N = 0$.

Theorem 7: Every irreducible module over a nil-semisimple ring with DCC on right ideals is isomorphic to a minimal right ideal of R.
Proof: Let $0 \neq m \in M$. Then $M = mR$ so $R/A(m) \cong M$.
A(m) is a right ideal and so is a unitary submodule of R.
So $R = A(m) \oplus S$ for some submodule $S \cong R/A(m) \cong M$.
Since M is irreducible, S is a minimal right ideal. ✋☺

Theorem 8: If R is a direct sum of matrix rings, R_i, over division rings then:
(1) every minimal right ideal of R is a minimal right ideal of some R_i.
(2) two minimal right ideals are isomorphic if and only if they come from the same R_i.
Proof: (1) Let S be a minimal right ideal of R.
Then for each i, $SR_i = 0$ or S.
For at least one i we have $SR_i = S$, whence $S \subseteq R_i$ (and hence for exactly one i).
Since R_i is a direct summand of R, S is a right ideal of R_i.

Let $S \subseteq R_r$ and $T \subseteq R_s$ be minimal right ideals.
Suppose $r \neq s$.
Then $ST = 0$.
If $f:S\to T$ is an R-module isomorphism and $x \in S$, $y\in T$ then $0 = (xy)f = (xf)y$.
Thus $T^2 = 0$ and so $(RT)(RT) \subseteq RT^2 = 0$.
By semi-simplicity the 2-sided ideal $RT = 0$, a contradiction.

Now suppose $r = s$.
We observe that the minimal right ideals of $M_n(D)$ are of the form:

$$S_r = \{(a_{ij}) \mid a_{ij} = 0 \text{ if } i \neq r\}.$$

and that the map $A\to E_{ji}A$ is an isomorphism from S to T.
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